Covariant Renormalizable Modified and Massive Gravity Theories on (Non) Commutative Tangent Lorentz Bundles

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Abstract

The fundamental field equations in modified gravity (including general relativity; massive and bimetric theories; Hořava-Lifshits, HL; Einstein–Finsler gravity extensions etc) posses an important decoupling property with respect to nonholonomic frames with 2 (or 3) +2+2+... spacetime decompositions. This allows us to construct exact solutions with generic off–diagonal metrics depending on all spacetime coordinates via generating and integration functions containing (un-) broken symmetry parameters. Such nonholonomic configurations/ models have a nice ultraviolet behavior and seem to be ghost free and (super) renormalizable in a sense of covariant and/or massive modifications of HL gravity. The apparent noncommutativity and breaking of Lorentz invariance by quantum effects can be encoded into fibers of noncommutative tangent Lorentz bundles for corresponding "partner" anisotropically induced theories. We show how the constructions can be extended to include conjectured covariant reonormalizable models with massive graviton fields and effective Einstein fields with (non)commutative variables.

Keywords: (non) commutative massive gravity, quantum gravity, covariant modifications of HL gravity.

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1 Introduction

As there is no hint how a quantum gravity theory (QG) can be formulated and verified for high energies, it is important to explore and compare certain promising models which can be unified into a unified commutative and noncommutative geometric formalism related to a general method of construction exact solutions. In this paper, we analyze three intensively developing approaches to quantum gravity. The aim is to show that we are able to formulate any such a quantum theory as a consistent, predictive and observational (in modern cosmology) one with phenomenological implications in high energy particle physics. The first approach is based on a proposal [1] (the so–called Hořava – Lifshitz, HL, gravity) to consider Lorentz non–invariant theories with scaling properties of space, x^i, and time, t, coordinates re–parameterized in the form $(\mathbf{x}, t) \rightarrow (b\mathbf{x}, b^z t)$, where $z = 2, 3, ...$ This modifies the ultraviolet (UV) behavior of the graviton propagator by changing $1/|\mathbf{k}|^2 \rightarrow 1/|\mathbf{k}|^{2z}$, where $\mathbf{k}$ is the spacial momentum. Such constructions were originally performed with the lack of full diffeomorphysm invariance and resulted in the impossibility to exclude completely certain un–physical modes, see critical remarks in a series of works [2]. Latter, a covariant renormalizable gravity model developing a HL–like gravity to full diffeomorphysm invariance was constructed [3]. The main idea was to broke the Lorentz–invariance of the graviton propagator by introducing a non–standard coupling with an unknown fluid. Such theories with possible physical transverse modes seem to be (super–) renormalizable and certain applications in modern accelerating cosmology where provided.

The second approach is related to a recent substantial progress made with massive/ bimetric gravity theories [4] (for historical remarks, motivations and review of results and applications, see also [5] and references therein). There are involved two metrics (for certain models, it is considered also a second connection) when the second one describe an effective exotic matter induced by massive gravitons. Such theories do not suffer from the ghost instability in a well defined perturbation theory (away from the decoupling limit). In a more general context (third approach), a QG must include noncommutative type configurations, for instance, induced by Schrödinger type uncertainty relations with the Planck quantum constant [6]. We can encode such theories into certain geometric models on (co) tangent bundles to spacetime manifolds (in standard form, there are used Lorentz manifolds and/or various modifications with nontrivial torsion, extra dimensions etc). It is possible to consider commutative and noncommutative modified gravity theories admitting exact off–diagonal solutions [7] with classical and quantum variables [8] subjected to nonholonomic constraints and with possible generalizations to Finsler–Lagrange–Hamilton variables [9].

In a series of works [7, 6, 9], there were provided detailed proofs and examples when the gravitational field eqs in general relativity, GR, and var-

ious modifications (with commutative and noncommutative variables, extra dimensions, massive terms, modified Lagrange densities, anistoropic dependencies on velocity/momentum type variables, string like and brane theories etc) can be decoupled with respect to certain classes of nonholonomic frames. The solutions for such nonlinear systems of (generalized/modified) Einstein equations can be constructed in very general forms. Such spacetimes are characterized by the generic off–diagonal metrics which can not be diagonalized via coordinate transforms and depend on all spacetime coordinates via corresponding generating and integration functions and possible (broken, or preserving) symmetry parameters. Possible nontrivial torsion configurations can be nonholonomically constrained to the Levi–Civita ones (with zero torsion). Choosing necessary types of generating/ integration functions and parameters and performing corresponding deformations of frame, metric and connection structures, we can model effective nonlinear interactions, (modified) massive gravity effects, with scaling properties and local anisotropies, which can be renormalizable.

In this work, we study (non) commutative massive gravity theories which can equivalently modelled as Lagrange density modified ones and encoded in (effective) Einstein spaces and generalizations on tangent Lorentz bundles. We shall state the conditions when (effective/modified) Einstein equations transform nonlinearly, by imposing corresponding classes of nonholonomic constraints, to nonlinear systems of partial differential eqs (PDE) with parametric dependence of solutions which under quantization "survive" and stabilize in some rescaled/anisotropic and renormalized forms.

The paper is organized as follows. In section 2, we state the actions for modelling commutative modified and effective theories of gravity on Lorentz manifolds. There are provided the corresponding generalized gravitational field equations. We consider also nonholonomic deformations of such models on (non) commutative tangent bundles determined by Schödinger type complex/noncommutative relations. Section 3 is devoted to a geometric method for decoupling and integrating the gravitational field equations with respect to nonholnomic frames. Then we consider off–diagonal solutions mimicking such effective theories for nonstandard perfect fluid coupling in section 4. We speculate how modified (non) commutative massive gravity theories can be renormalize in a covariant HL sense using effective Einstein and/or Einstein–Finsler type spaces. Finally, we conclude the paper in section 5.

2 (Non) Commutative Modified Massive Gravity

2.1 Actions for equivalent commutative gravity theories

We consider four equivalent models determined by actions

$$S = \frac{1}{16\pi} \int \delta u^4 \sqrt{|\mathbf{g}_{\alpha\beta}|}\ \mathcal{L},\ \text{ for } \mathcal{L} = {}^{[i]}\mathcal{L}, [i] = 1,2,3,4; \qquad (1)$$

(two f–modified, a massive and an effective Einstein gravity theories), where

$$
\begin{aligned}
{}^{[1]}\mathcal{L} &= \widehat{f}(\widehat{R}) - \frac{\mathring{\mu}^2}{4}\mathcal{U}(\mathbf{g}_{\mu\nu}, \mathbf{K}_{\alpha\beta}) + {}^{m}L,\ {}^{[2]}\mathcal{L} = {}^{s}\widetilde{R} + \widetilde{L}, \\
{}^{[3]}\mathcal{L} &= R + L(T^{\mu\nu}, R_{\mu\nu}),\ {}^{[4]}\mathcal{L} = f(\check{R}) + {}^{m}L.
\end{aligned}
$$

Such theories are modelled on a four dimensional (4d) pseudo–Riemannian manifold $\mathbf{V}$ with physical metric $\mathbf{g} = \{\mathbf{g}_{\mu\nu}\}$ of signature $(+,+,+,-)$. In GR, $\mathbf{V}$ is defined as a Lorentz manifold with a corresponding axiomatic and physical interpretation which can be extended to Einstein – Finsler like models on tangent Lorentz bundles [9]. Commutative modifications of gravity theories are considered for the same metric structure but with different connections and/or Lagrange densities. For massive gravity models with "small" graviton mass $\mathring{\mu}$, the physical spacetime $\mathbf{V}$ is modelled in a similar form but with an additional bimetric structure related to a "sophisticate" potential of graviton $\mathcal{U}$, see details in [4]; it is involved a fiducial metric determined by a tensor field $\mathbf{K}_{\alpha\beta}$, see details in Refs. [4]. The symbols ${}^{m}L$ and L are used respectively for Lagrange densities of matter fields and effective matter.

In above formulas, $\widehat{R}$ is the scalar curvature for an auxiliary (canonical) connection $\widehat{\mathbf{D}}$ uniquely determined by $\mathbf{g}$ for a conventional nonholonomic horizontal, h, and vertical, v, splitting following two conditions: 1) It is metric compatible, $\widehat{\mathbf{D}}\mathbf{g} = 0$, and 2) its h- and v-torsions are zero (but there are nonzero $h-v$ components of torsion $\widehat{\mathcal{T}}$ completely determined by $\mathbf{g}$). Geometrically such a decomposition (splitting) is stated as a Whitney sum

$$
\mathbf{N} : T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}, \tag{2}
$$

see details in [7].[1] The "priority" of the connection $\widehat{\mathbf{D}}$ is that it allows to decouple the field equations in various gravity theories which allows us to construct exact solutions in very general off–diagonal forms. Using distortion relations $\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}[\widehat{\mathcal{T}}]$ we can recover configurations with the Levi–Civita (LC) connection ∇ which together with the distorting tensor $\widehat{\mathbf{Z}}$, and $\widehat{\mathbf{D}}$ are completely defined by $\widehat{\mathcal{T}}$ (for such models, by $(\mathbf{g}, \mathbf{N})$). Having constructed integral varieties of some gravitational field equations in terms of $\widehat{\mathbf{D}}$, we can impose additional nonholonomic (non–integrable constraints) when $\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}=0} \rightarrow \nabla$ and $\widehat{R} \rightarrow R$, where R is the scalar curvature of ∇. As a result, it is possible to extract generic off–diagonal solutions in GR and/or other theories with ∇.

[1] For a conventional $2+2$ splitting, the coordinates can be labelled in the form $u^{\alpha} = (x^i, y^a)$, or $u = (x, y)$, with indices $i, j, k... = 1, 2$ and $a, b, ... = 3, 4$. Boldface symbols are used in order to emphasize that certain geometric/physical objects and/or formulas are written in a N–adapted form. There will be considered left up/low indices as labels for certain classes of geometric/physical objects. We shall use the Einstein rule on summation of repeating right up-low indices if the contrary will be not stated.

All terms in actions (1) are stated by the same metric structure $\mathbf{g} = \{g_{\alpha\beta}\}$ for standard and/or modified models of gravity theory generated by different (effective) Lagrangians when curvature scalars $(\widehat{R}, \widetilde{R}, R, \breve{R})$, for necessary type linear connections, effective cosmological constant $\widehat{\Lambda}$, non–standard coupling of Ricci, $R_{\mu\nu}$, and energy–momentum, $T^{\mu\nu}$, tensors etc. A geometric model with ${}^{[1]}\mathcal{L}$ can be used for constructing general classes of generic off–diagonal solutions in GR and modifications (including massive gravity). Such a theory models effects with broken Lorentz invariance, non–standard effective anistoropic fluid coupling and behavior of the polarized propagator in the ultraviolet/infrared region. The solutions for ${}^{[2]}\mathcal{L}$ will be used as "bridges" between generating functions determining certain classes of generic off–diagonal Einstein manifolds and effective models with anisotropies, massive effects and resulting violations for Lorentz symmetry. The effective Lagrange density ${}^{[3]}\mathcal{L}$ is similar to that a covariant renormalization gravity as in [3]. For corresponding conditions on f, the value ${}^{[4]}\mathcal{L}$ is related to modified theories. In almost Kähler variables, theories of type ${}^{[1]}\mathcal{L}$– ${}^{[3]}\mathcal{L}$ can be quantized using non–perturbative methods for deformation quantization and A–brane quantization, or as perturbative gauge like models [8].

In this paper, we study modified gravity models when the spacetime metrics for theories derived from ${}^{[3]}\mathcal{L}$ and/or ${}^{[4]}\mathcal{L}$ are approximated ${}^{\diamond}\mathbf{g}_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ for perturbative models with a flat background metric $\eta_{\alpha\beta}$. Such solutions are with massive gravity effects, breaking of Lorentz symmetry and, for well stated conditions, result into effective models with covariant renormalization. Using frame transforms $\mathbf{g}_{\alpha\beta} = e^{\alpha'}_{\ \alpha} e^{\beta'}_{\ \beta}\ {}^{\diamond}\mathbf{g}_{\alpha'\beta'}$, we shall connect the classical and quantum perturbative solutions to the theories (and their nonlinear/generalized solutions) determined by ${}^{[1]}\mathcal{L}$, or ${}^{[2]}\mathcal{L}$ and background configurations for corresponding metric and connection structure. Via nonholonomic constraints, the physical effects are derived to be similar to those for some (super-) renormalizable theories ${}^{[3]}\mathcal{L}$ and/or ${}^{[4]}\mathcal{L}$.

2.2 Nonholonomic distributions and noncommutative uncertainty relations

A class of (quantum type) modified noncommutative gravity theories are physically motivated by Schrödinger type uncertainty relations $\hat{u}^{\alpha}\hat{p}^{\beta} - \hat{p}^{\beta}\hat{u}^{\alpha} = i\hbar\widehat{\theta}^{\alpha\beta}$, where $\hbar$ is the Planck constant, $i^2 = -1$, and $\hat{u}^{\alpha}$ and $\hat{p}^{\beta}$ are, respectively, certain coordinate and momentum type operators. Noncommutative geometry/ gravity models are elaborated on "θ–extensions" of tangent Lorentz bundles $T\mathbf{V} \to {}^{\theta}T\mathbf{V}$, (there are used also co–tangent bundles $T^{*}\mathbf{V} \to {}^{\theta}T^{*}\mathbf{V}$ etc). Such extensions are determined by noncommutative complex distributions stated by "generalized uncertainty" relations

$$u^{\alpha_s}u^{\beta_s} - u^{\beta_s}u^{\alpha_s} = i\theta^{\alpha_s\beta_s}, \quad (3)$$

where the antisymmetric matrix $\theta = (\theta^{\alpha_s \beta_s})$ can be taken with constant coefficients with respect to certain frame of reference on "small" θ–deformations $\mathbf{V} \rightarrow {}^{\theta}\mathbf{V}$. The label "$s$" is considered for three two dimensional (2-d) "shells" $s = 0, 1, 2,$ for a conventional splitting on ${}^{\theta}T\mathbf{V}$, $\dim({}^{\theta}T\mathbf{V}) = 4 + 2s = 2+2+2+2 = 8$. The coordinates and indices are respectively parameterized $u^{\alpha_s} = (x^{i_s}, y^{a_s})$. We write $u^{\alpha} = (x^i, y^a)$ and consider for

$$
\begin{aligned}
s &= 0 : u^{\alpha_0} = (x^{i_0}, y^{a_0}) = u^{\alpha} = (x^i, y^a); \\
s &= 1 : u^{\alpha_1} = (x^{\alpha} = u^{\alpha}, y^{a_1}) = (x^i, y^a, y^{a_1}); \\
s &= 2 : u^{\alpha_2} = (x^{\alpha_1} = u^{\alpha_1}, y^{a_2}) = (x^i, y^a, y^{a_1}, y^{a_2}); \\
s &= 3 : u^{\alpha_3} = (x^{\alpha_2} = u^{\alpha_2}, y^{a_3}) = (x^i, y^a, y^{a_1}, y^{a_2}, y^{a_3}), ...,
\end{aligned} \tag{4}
$$

when indices run corresponding values $i, j, ... = 1, 2; a, b, ... = 3, 4; a_1, b_1 ... = 5, 6; a_2, b_2 ... = 7, 8;\ a_3, b_3 ... = 9, 10, ...$ and, for instance, $i_1, j_1, ... = 1, 2, 3, 4;$ $i_2, j_2, ... = 1, 2, 3, 4, 5, 6;\ i_3, j_3, ... = 1, 2, 3, 4, 5, 6, 7, 8; ...$ In brief, we shall write $u = (x, y);\ {}^1u = (u,\ {}^1y) = (x, y,\ {}^1y),\ {}^2u = (\ {}^1u,\ {}^2y) = (x, y,\ {}^1y,\ {}^2y), ...$

In this work, the geometric objects with $s > 0$ are some θ–deformations determined by distributions of type (3) and additional assumptions on "inner products", N–adapted connections etc. We shall follow a geometric principle that for elaborating noncommutative (quantum) geometric models there are considered classical nonlinear configurations (exact solutions) for a gravity theory (1) and then elaborated "small" deformations to noncommutative gravity models with additional quantum parameters encoded for $s \neq 0$.

The noncommutative relations (3) and coordinate parameterizations (4) are adapted to Whitney sums[2] ${}^s\mathbf{N} : T\ {}^s\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \oplus\ {}^1v\mathbf{V} \oplus\ {}^2v\mathbf{V}$. This prescribes a local fibered structure on ${}^{\theta}\mathbf{V}$, when the coefficients of N–connection, $N^{a_s}_{i_s}$, for ${}^s\mathbf{N} = N^{a_s}_{i_s}(\ {}^su, \theta) dx^{i_s} \otimes \partial/\partial y^{a_s}$, and states a system of N–adapted local bases with N-elongated partial derivatives, $\mathbf{e}_{\nu_s} = (\mathbf{e}_{i_s}, e_{a_s})$, and cobases with N–adapted differentials, $\mathbf{e}^{\mu_s} = (e^{i_s}, \mathbf{e}^{a_s})$. For a 4-d commutative Lorentz manifold $\mathbf{V}$,

$$
\mathbf{e}_i = \frac{\partial}{\partial x^i} - \ N^a_i \frac{\partial}{\partial y^a},\ e_a = \frac{\partial}{\partial y^a}, \tag{5}
$$

$$
e^i = dx^i, \mathbf{e}^a = dy^a + \ N^a_i dx^i, \tag{6}
$$

and on $s = 1, 2$ shells, (if any $\theta^{\alpha_s \beta_s} \neq 0$, the coefficients depend on coordinates and such parameters, for instance, $N^{a_s}_{i_s}(\ {}^1u,\ {}^2y, \theta)$),

$$
\begin{aligned}
\mathbf{e}_{i_s} &= \frac{\partial}{\partial x^{i_s}} - \ N^{a_s}_{i_s} \frac{\partial}{\partial y^{a_s}},\ e_{a_s} = \frac{\partial}{\partial y^{a_s}}, & (7) \\
e^{i_s} &= dx^{i_s}, \mathbf{e}^{a_s} = dy^{a_s} + \ N^{a_s}_{i_s} dx^{i_s}. & (8)
\end{aligned}
$$

The N–adapted operators (5) and (7) satisfy certain anholonomy relations

$$
[\mathbf{e}_{\alpha_s}, \mathbf{e}_{\beta_s}] = \mathbf{e}_{\alpha_s} \mathbf{e}_{\beta_s} - \mathbf{e}_{\beta_s} \mathbf{e}_{\alpha_s} = W^{\gamma_s}_{\alpha_s \beta_s} \mathbf{e}_{\gamma_s}, \tag{9}
$$

[2]in certain geometric and physical theories [7, 6], it is used the term nonlinear connection, N–connection

completely defined by the N–connection coefficients and their partial derivatives, $W^{b_s}_{i_s a_s} = \partial_{a_s} N^{b_s}_{i_s}$ and $W^{a_s}_{j_s i_s} = \Omega^{a_s}_{i_s j_s}$, where the curvature of N–connection is $\Omega^{a_s}_{i_s j_s} = \mathbf{e}_{j_s}\left(N^{a_s}_{i_s}\right) - \mathbf{e}_{i_s}\left(N^{a_s}_{j_s}\right)$. The quantum noncommutative structure is encoded into such nonholonomic distributions. The geometric objects with coefficients defined with respect to N–adapted frames are called respectively distinguished metrics, distinguished tensors etc (in brief, d–metrics, d–tensors etc)

Any metric structure ${}^{s}\mathbf{g} = \{\mathbf{g}_{\alpha_s \beta_s}\}$ on ${}^{\theta}\mathbf{V}$ with $\theta = (\theta^{\alpha_s \beta_s})$ can be written as a distinguished metric (d–metric)

$$\begin{aligned} {}^{s}\mathbf{g} &= g_{i_s j_s}({}^{s}u, \theta)\ e^{i_s} \otimes e^{j_s} + g_{a_s b_s}({}^{s}u, \theta) \mathbf{e}^{a_s} \otimes \mathbf{e}^{b_s} \\ &= g_{ij}(x)\ e^{i} \otimes e^{j} + g_{ab}(u)\ \mathbf{e}^{a} \otimes \mathbf{e}^{b} + \\ &\quad g_{a_1 b_1}({}^{1}u, \theta)\ \mathbf{e}^{a_1} \otimes \mathbf{e}^{b_1} + g_{a_2 b_2}({}^{2}u, \theta) \mathbf{e}^{a_2} \otimes \mathbf{e}^{b_2}. \end{aligned} \tag{10}$$

In coordinate frames, (10) is parameterized equivalently by generic off–diagonal matrices (which can not be diagonalized via coordinate transforms),

$${}^{s}\mathbf{g} = g_{\alpha_s \beta_s} e^{\alpha_s} \otimes e^{\beta_s} = g_{\underline{\alpha}_s \underline{\beta}_s} du^{\underline{\alpha}_s} \otimes du^{\underline{\beta}_s},\ s = 0, 1, 2, ..., \tag{11}$$

where coefficients transform and $g_{\alpha_s \beta_s} = e^{\underline{\alpha}_s}_{\alpha_s} e^{\underline{\beta}_s}_{\beta_s} g_{\underline{\alpha}_s \underline{\beta}_s}$, for respective frames and local coordinate bases $e_{\alpha_s} = e^{\underline{\alpha}_s}_{\alpha_s}({}^{s}u)\partial/\partial u^{\underline{\alpha}_s}, \partial_{\beta_s} := \partial/\partial u^{\beta_s}$, when

$$\underline{g}_{\alpha_s \beta_s}({}^{s}u) = \begin{bmatrix} g_{i_s j_s} + h_{a_s b_s} N^{a_s}_{i_s} N^{b_s}_{j_s} & h_{a_s e_s} N^{e_s}_{j_s} \\ h_{b_s e_s} N^{e_s}_{i_s} & h_{a_s b_s} \end{bmatrix}, s = 0, 1, 2.$$

The metrics (10) and/or (11) encode higher shells dependencies on noncommutative parameters $\theta = (\theta^{\alpha_s \beta_s})$ and distributions (3). A self–consistent approach to such theories is based on the Groenewold–Moyal product (star product, or $\star$–product) inspired by the foundations of quantum mechanics. We apply a formalism elaborated in [10] but modified for nonholonomic distributions and connections with $\nabla \to \mathbf{D} = \tilde{\mathbf{D}}$, or $= \widehat{\mathbf{D}}$, and almost Kähler variables determined naturally by data $(\mathbf{g}, \mathbf{N})$, see [6]. The constructions are based on formal power series $C^{\infty}(\mathbf{V})[[\ell]]$ with deformation parameter $\ell = i\hbar$, where $i^2 = -1$ and where $\hbar = h/2\pi$ is used for respective Plank constants. The symbol $\tilde{\mathbf{D}}$ is used for the Cartan connection which is adapted by a canonical N–connection structure $\tilde{\mathbf{N}}$ completely determined by regular effective Lagrange generating function $\mathcal{L}(x, y)$. Such a value can be always prescribed on a (pseudo) Riemannian spacetime $\mathbf{V}$ and modelled on $T\mathbf{V}$ following the principle that the semi–spray equations are equivalent to the Euler–Lagrange ones, see details in [9].

There are induced by any $\mathcal{L}(x, y)$ other important geometric structures. We remember that an almost complex structure is a linear operator $\mathbf{J}$ acting on vectors on $T\mathbf{V}$ via actions on N–adapted frames, $\mathbf{J}(\mathbf{e}_i) = -e_{2+i}$ and $\mathbf{J}(e_{2+i}) = \mathbf{e}_i$, where $\mathbf{J} \circ \mathbf{J} = -\mathbb{I}$, for $\mathbb{I}$ being the unity matrix. If such a

structure is canonical, we can write $\tilde{\mathbf{J}}$ for $\mathbf{N} = \tilde{\mathbf{N}}$. A canonical almost Kähler space with $\mathbf{g} = \tilde{\mathbf{g}}$, $\mathbf{N} = \tilde{\mathbf{N}}$ and $\mathbf{J} = \tilde{\mathbf{J}}$ canonically defined by $\mathcal{L}$, when $\tilde{\theta}(\cdot,\cdot) := \tilde{\mathbf{g}}\left(\tilde{\mathbf{J}}\cdot,\cdot\right)$. We can introduce $\tilde{\theta} = d\tilde{\omega}$ for $\tilde{\omega} := \frac{1}{2}\frac{\partial \mathcal{L}}{\partial y^i}dx^i$, where $d\tilde{\theta} = dd\tilde{\omega} = 0$. If θ is related to a canonical $\tilde{\theta}$ via frame transforms, $\theta_{\alpha'\beta'}e^{\alpha'}_{\ \alpha}e^{\beta'}_{\ \beta} = \tilde{\theta}_{\alpha\beta}$, we positively construct an almost Kähler structure. Fixing a convenient $\mathcal{L}$ (with a corresponding N–splitting $\tilde{\mathbf{N}}$), we generate equivalent geometric models of nonholonomic manifolds with $(\mathbf{g},\mathbf{N}) \approx (\tilde{\mathbf{g}},\tilde{\mathbf{N}}) \approx (\theta,\mathbf{J}) \approx (\tilde{\theta},\tilde{\mathbf{J}})$. The models with shell d–metric, ${}^{s}\mathbf{g}$, and almost symplectic, ${}^{s}\theta = \{\theta^{\mu_s\nu_s}\}$, are convenient for constructing exact solutions and/or study Finsler–Lagrange geometries but the almost symplectic ones can be used for noncommutative geometry with distributions of type (3) and/or deformation quantization. For simplicity, we shall omit the left label "s" if that will not result in ambiguities.

The canonical (Cartan) covariant star product ${}^{\theta}T\mathbf{V}$ is introduced as

$$\alpha\tilde{\star}\beta := \sum_k \frac{\ell^k}{k!}\theta^{\mu_1\nu_1}...\theta^{\mu_k\nu_k}(\mathbf{D}_{\mu_1}\ldots\mathbf{D}_{\mu_k})\cdot(\mathbf{D}_{\nu_1}\ldots\mathbf{D}_{\nu_k}), \qquad (12)$$

for $\mathbf{D} = \tilde{\mathbf{D}}$ (or $= \widehat{\mathbf{D}}$). The product $\tilde{\star}$ is adapted to a N–connection structure (2) and maps d–tensors into d–tensors. For $\mathbf{D} \to \nabla$, this operator transforms into similar noncommutative generalizations of the (pseudo) Riemann geometry if θ is fixed for a symplectic manifold, $\tilde{\star} \to \star$. It is possible to define s–shell associative star operators ${}^{s}\tilde{\star}$ if $\mathbf{D}_{\mu_s} = (\mathbf{D}_{i_s},\mathbf{D}_{a_s})$,

$$\alpha\ ({}^{s}\tilde{\star})\beta = \sum_k \frac{\ell^k}{k!}\theta^{a_1b_1}...\theta^{a_kb_k}(\mathbf{D}_{a_1}\ldots\mathbf{D}_{a_k})\cdot(\mathbf{D}_{b_1}\ldots\mathbf{D}_{b_k}).$$

The star product (12) can be re–expressed in the form

$$\alpha\tilde{\star}\beta := \alpha\beta + \sum_k^{\infty}\ell^k\mathbf{C}_k(\alpha,\beta), \qquad (13)$$

where the bilinear operators $\mathbf{C}_k$ are N–adapted, i.e. d–operators.

The product $\tilde{\star}$ (13) satisfies such properties:

1. associativity, $\alpha\tilde{\star}(\beta\tilde{\star}\gamma) = (\alpha\tilde{\star}\beta)\tilde{\star}\gamma$;

2. it is defined the Poisson bracket, $\mathbf{C}_1(\alpha,\beta) = \{\alpha,\beta\} = \theta^{\mu_s\nu_s}\mathbf{D}_{\mu_s}\alpha\cdot\mathbf{D}_{\nu_s}\beta$, distorted by the nonholonomically induced torsion completely defined by the metric (almost symplectic) structure; here we note antisymmetry, $\{\alpha,\beta\} = -\{\beta,\alpha\}$, and the Jacoby identity,

$$\{\alpha,\{\beta,\gamma\}\} + \{\gamma,\{\alpha,\beta\}\} + \{\beta,\{\alpha,\gamma\}\} = 0;$$

3. there is an N–adapted stability of type $\alpha\tilde{\star}\beta = \alpha\cdot\beta$ if ${}^{s}\mathbf{D}\alpha = 0$ or ${}^{s}\mathbf{D}\beta = 0$;

4. the Moyal symmetry, $\mathbf{C}_k(\alpha,\beta)=(-1)^k\mathbf{C}_k(\beta,\alpha)$;

5. the N-adapted derivation with Leibniz rule,

$$\begin{aligned} {}^s\mathbf{D}(\alpha\tilde{\star}\beta) &= ({}^s\mathbf{D}\alpha)\tilde{\star}\beta+\alpha\tilde{\star}({}^s\mathbf{D}\beta) \\ &= ((h\mathbf{D}+{}^s v\mathbf{D})\alpha)\,\tilde{\star}\beta+\alpha\tilde{\star}((h\mathbf{D}+{}^s v\mathbf{D})\beta). \end{aligned}$$

For applications in quantum physics, it is important the Hermitian property, $\overline{\alpha\tilde{\star}\beta}=\overline{\beta}\tilde{\star}\overline{\alpha}$, with complex conjugation "-", when $\mathbf{g}_{\alpha_s\beta_s}=\frac{1}{2}\left(\overline{\mathbf{e}}_{\alpha_s}\tilde{\star}\mathbf{e}_{\beta_s}+\overline{\mathbf{e}}_{\beta_s}\tilde{\star}\mathbf{e}_{\alpha_s}\right)$. It should be noted that because ${}^s\tilde{\mathbf{D}}\tilde{\theta}=0$, we can write $\theta^{\mu_s\nu_s}\tilde{\star}\chi=\theta^{\mu_s\nu_s}\cdot\chi$.

Using canonical almost symplectic data $(\tilde{\star},\ {}^s\tilde{\mathbf{D}})$, it is possible to elaborate an associative star product calculus in noncommutative spacetime which is completely defined by the metric structure in N–adapted form and keeps the covariant property. We derive a N–adapted local frame structure $\tilde{\mathbf{e}}_{\alpha_s}=(\mathbf{e}_{i_s},\tilde{\mathbf{e}}_{a_s})$ which can be constructed by noncommutative deformations of $\mathbf{e}_\alpha$,

$$\begin{aligned} \tilde{\mathbf{e}}_{\alpha_s}^{\ \underline{\alpha}_s} &= \mathbf{e}_{\alpha_s}^{\ \underline{\alpha}_s}+i\theta^{\gamma_s\beta_s}\mathbf{e}_{\alpha_s\ \gamma_s\beta_s}^{\ \underline{\alpha}_s}+\theta^{\gamma_s\beta_s}\theta^{\tau_s\mu_s}\mathbf{e}_{\alpha_s\ \gamma_s\tau_s\mu_s}^{\ \underline{\alpha}_s}+\mathcal{O}(\theta^3), \qquad (14)\\ \tilde{\mathbf{e}}_{\star\underline{\alpha}_s}^{\alpha_s} &= \mathbf{e}_{\alpha_s}^{\ \underline{\alpha}_s}+i\theta^{\gamma_s\beta_s}\mathbf{e}_{\underline{\alpha}_s\gamma_s\beta_s}^{\alpha_s}+\theta^{\gamma_s\beta_s}\theta^{\tau_s\mu_s}\mathbf{e}_{\underline{\alpha}_s\gamma_s\beta_s\tau_s\mu_s}^{\alpha_s}+\mathcal{O}(\theta^3), \end{aligned}$$

subjected to the condition $\tilde{\mathbf{e}}_{\star\underline{\alpha}_s}^{\alpha_s}\star\tilde{\mathbf{e}}_{\alpha_s}^{\ \underline{\beta}_s}=\delta_{\underline{\alpha}_s}^{\underline{\beta}_s}$, where $\delta_{\underline{\alpha}}^{\underline{\beta}}$ is the Kronecker tensor. The values $\mathbf{e}_{\alpha_s\ \gamma_s\beta_s}^{\ \underline{\alpha}_s}$ and $\mathbf{e}_{\alpha_s\ \gamma_s\tau_s\mu_s}^{\ \underline{\alpha}_s}$ can be written in terms of $\mathbf{e}_{\alpha_s}^{\ \underline{\alpha}_s},\theta^{\gamma_s\beta_s}$ and the spin distinguished connection corresponding to ${}^s\tilde{\mathbf{D}}$, or ${}^s\widehat{\mathbf{D}}$, see similar formulas in [6, 10].

The noncommutative deformations of a metric, $\mathbf{g}\to{}^s\mathbf{g}$, can be defined and computed ${}^s\mathbf{g}_{\alpha_s\beta_s}=\frac{1}{2}\eta_{\underline{\alpha}_s\underline{\beta}_s}\left[\tilde{\mathbf{e}}_{\alpha_s}^{\ \underline{\alpha}_s}\star\left(\tilde{\mathbf{e}}_{\beta_s}^{\ \underline{\beta}_s}\right)^+ +\tilde{\mathbf{e}}_{\beta_s}^{\ \underline{\beta}_s}\star\left(\tilde{\mathbf{e}}_{\alpha_s}^{\ \underline{\alpha}_s}\right)^+\right]$, where $(\ldots)^+$ is used for the Hermitian conjugation and $\eta_{\underline{\alpha}_s\underline{\beta}_s}$ denotes the flat Minkowski spacetime metric extended on $T\mathbf{V}$. We can parameterize the noncommutative and nonholonomic transforms when ${}^s\mathbf{g}_{\alpha_s\beta_s}({}^s u,\theta)$ (10) is with real coefficients which for $s=1,2$ depend only on even powers of θ,

$$\begin{aligned} g_i(u) &= \grave{g}_i(x^k),\ h_a=\grave{h}_a(u),\ h_{a_s}=\grave{h}_{a_s}(u)+{}^2h_{a_s}(u)\theta^2+\mathcal{O}(\theta^4), \\ N_{i_s}^{a_s}({}^s u,\theta) &= \grave{N}_{i_s}^{a_s}({}^s u)+{}^2N_{i_s}^{a_s}({}^s u)\theta^2+\mathcal{O}(\theta^4). \qquad (15) \end{aligned}$$

This allows us to treat θ as some integration parameters related to superposition of Killing symmetries and anholonomic frame transformations [11]. For simplicity, we shall not write θ in explicit form if that will not result in ambiguities for any notation of type $N_{i_s}^{a_s}({}^s u,\theta), N_{i_s}^{a_s}(\theta)$, or $N_{i_s}^{a_s}({}^s u)$.

3 Decoupling & Integration of (Non)Commutative Modified Massive Gravity

3.1 Effective Einstein equations

Applying a N–adapted variational calculus for ${}^{[1]}\mathcal{L}$ on $\mathbf{V}$, see details in [7, 4, 5], we derive the equations of motion for 4–d modified massive gravity

$$(\partial_{\widehat{R}}\widehat{f})\widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\widehat{f}(\widehat{R})\mathbf{g}_{\alpha\beta} + \mathring{\mu}^2\mathbf{X}_{\alpha\beta} = M_{Pl}^{-2}\mathbf{T}_{\alpha\beta}, \quad (16)$$

where M_{Pl} is the Plank mass, $\widehat{\mathbf{R}}_{\mu\nu}$ is the Einstein tensor for a pseudo–Riemannian metric $\mathbf{g}_{\mu\nu}$ and $\widehat{\mathbf{D}}$, $\mathbf{T}_{\mu\nu}$ is the standard energy–momentum tensor for matter. The effective energy–momentum tensor $\mathbf{X}_{\mu\nu}$ is determined by the potential of graviton $\mathcal{U} = \mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4$, where α_3 and α_4 are free parameters and $\mathcal{U}_2, \mathcal{U}_3$ and $\mathcal{U}_4$ are certain polynomials on traces of some other polynomials of the matrix $\mathcal{K}_\mu^\nu = \delta_\mu^\nu - \left(\sqrt{g^{-1}\Sigma}\right)_\mu^\nu$. There are involved four Stückelberg fields $\phi^{\underline{\mu}}$ as $\Sigma_{\mu\nu} = \partial_\mu\phi^{\underline{\mu}}\partial_\nu\phi^{\underline{\nu}}\eta_{\underline{\mu\nu}}$, when $\eta_{\underline{\mu\nu}} = (1,1,1,-1)$. A parameter choice $\alpha_3 = (\alpha-1)/3, \alpha_4 = (\alpha^2-\alpha+1)/12$ is optimal if we want to avoiding potential ghost instabilities and $\mathbf{X}_{\mu\nu} = \alpha^{-1}\mathbf{g}_{\mu\nu}$. It is possible to find exact off–diagonal solutions of (16) if we fix the coefficients $\{N_i^a\}$ of $\mathbf{N}$ and local frames for $\widehat{\mathbf{D}}$ when $\widehat{R} = const$ and $\partial_\alpha\widehat{f}(\widehat{R}) = (\partial_{\widehat{R}}\widehat{f}) \times \partial_\alpha\widehat{R} = 0$. In general, $\partial_\alpha R \neq 0$ and $\partial_\alpha f(\breve{R}) \neq 0$. For simplicity, we shall consider configurations withe energy momentum sources $\mathbf{T}_{\mu\nu}$ and effective $\mathbf{X}_{\mu\nu}$ which (using frame transforms) can be parameterized with respect to N–adapted frames (5) and (6) in the form

$$\Upsilon_\beta^\alpha = \frac{1}{M_{Pl}^2(\partial_{\widehat{R}}\widehat{f})}(\mathbf{T}_\beta^\alpha + \alpha^{-1}\mathbf{X}_\beta^\alpha) = \frac{1}{M_{Pl}^2(\partial_{\widehat{R}}\widehat{f})}({}^mT + \alpha^{-1})\delta_\beta^\alpha = (\hat{\Upsilon} + \mathring{\Upsilon})\delta_\beta^\alpha, \quad (17)$$

for constant values $\hat{\Upsilon} := M_{Pl}^{-2}(\partial_{\widehat{R}}\widehat{f})^{-1}\,{}^mT$ and $\mathring{\Upsilon} = M_{Pl}^{-2}(\partial_{\widehat{R}}\widehat{f})^{-1}\alpha^{-1}$.

All above constructions can be extended in a (non) commutative form to extra shells $s = 1, 2, ...$ via formal re–definition of indices for higher dimensions, adapting with respect to bases (7) and (8) generated by noncommutative nonholonomic distributions (3). Under very general assumptions, the effective source can be parameterized in the form

$$\mathbf{\Upsilon}_{\delta_s}^{\beta_s} = ({}^s\hat{\Upsilon} + {}^s\mathring{\Upsilon})\delta_{\delta_s}^{\beta_s}. \quad (18)$$

The sources with $s = 1$ and 2 are considered as certain effective ones on "fibers" with noncommutative variables. For the simplest models, they can associated to certain effective cosmological constants or stated to be zero. Geometrically, such sources can be defined as N–adapted lifts from the base to the total space. Quantum corrections to certain classes of solutions can be modelled via certain stochastic (Finsler like or generalized ones [9, 8]) corrections to generating functions and sources. Such $({}^s\hat{\Upsilon} + {}^s\mathring{\Upsilon})$–terms

encode via nonholonomic constraints and the canonical d–connection ${}^s\widehat{\mathbf{D}}$ various physically important information on possible modifications of the GR theory on certain Lorentz manifolds backgrounds and generalized (co) tangent commutative and/or noncommutative bundles. On ${}^\theta T\mathbf{V}$, modified gravitational field equations of type (16) are written

$$ {}^s\widehat{\mathbf{R}}_{\ \beta_s\delta_s} - \frac{1}{2}\mathbf{g}_{\beta_s\delta_s}\ {}^{sc}\widehat{R} = \mathbf{\Upsilon}_{\beta_s\delta_s}, \tag{19} $$

$$ \widehat{L}^{c_s}_{a_s j_s} = e_{a_s}(N^{c_s}_{j_s}),\ \widehat{C}^{i_s}_{j_s b_s} = 0,\ \Omega^{a_s}_{\ j_s i_s} = 0, \tag{20} $$

where sources $\mathbf{\Upsilon}_{\beta_s\delta_s}$ (18) for $s=0$ are formally defined as in GR but on (non) commutative fibers using (12) and (13). For commutative extra dimensions, we can take n $\mathbf{\Upsilon}_{\beta_s\delta_s} \rightarrow \varkappa T_{\beta_s\delta_s}$ for ${}^s\widehat{\mathbf{D}} \rightarrow {}^s\nabla$, with effective sources as canonical lifts from the base 4–d spacetime determined by distributions (3). In general, the solutions of (19) are with nonholonomically induced torsion. If the conditions (20) are satisfied, the nonholonomically induced torsion is constrained to be zero and we get the Levi–Civita, LC, connection.

There are two classes of (non) commutative theories with effective gravitational field equations of type (19) and (possible) constraints (20):

1. Models generated for $s \neq 0$ as "twisted" noncommutative products adapted to nonholonomic (complex and/or real) distributions. For $s=0$, the equations (19) are equivalent to (16). To derive such equations we can use any action (1) and elaborate corresponding massive, modified Einstein gravity theories with generalized connections.

2. There is a more general class of theories when (19) and (18) are constructed directly on $T\ {}^s\mathbf{V}$ and/or ${}^\theta T\mathbf{V}$ with arbitrary N–connections. We shall not study such theories in this works even, for instance, certain classes of solutions compactified / warped / trapped on base commutative spacetimes may play an important role in modified gravity theories (see, for instance, Finsler branes etc [9]).

3.2 N–adapted symmetries

The decoupling property of N–adapted equations (19) can be proved in a straightforward form for ${}^s\widehat{\mathbf{D}}$ and metrics of type

$$ \begin{aligned} {}^s\mathbf{g} &= g_i(x^k)dx^i \otimes dx^i + h_a(x^k,y^4)\mathbf{e}^a \otimes \mathbf{e}^b + \\ & h_{a_1}(u^\alpha, y^6, \theta)\ \mathbf{e}^{a_1} \otimes \mathbf{e}^{a_1} + h_{a_2}(u^{\alpha_1}, y^8, \theta)\ \mathbf{e}^{a_2} \otimes \mathbf{e}^{b_2}, \end{aligned} \tag{21} $$

$$ \begin{aligned} &\text{where } \mathbf{e}^a = dy^a + N^a_i dx^i, \text{ for } N^3_i = n_i(x^k,y^4), N^4_i = w_i(x^k,y^4); \\ &\mathbf{e}^{a_1} = dy^{a_1} + N^{a_1}_\alpha du^\alpha, \text{ for } N^5_\alpha = {}^1n_\alpha(u^\beta, y^6, \theta), N^6_\alpha = {}^1w_\alpha(u^\beta, y^6, \theta); \\ &\mathbf{e}^{a_2} = dy^{a_2} + N^{a_2}_{\alpha_1} du^{\alpha_1}, \text{ for } N^7_{\alpha_1} = {}^2n_{\alpha_1}(u^{\beta_1}, y^8, \theta), N^8_{\alpha_1} = {}^2w_\alpha(u^{\beta_1}, y^8, \theta). \end{aligned} \tag{22} $$

Such ansatz contains a Killing vector $\partial/\partial y^7$ because the coordinate y^7 is not contained in the coefficients of such metrics. If $\theta \rightarrow 0$, we generate

off–diagonal metrics of 4–d effective Einstein equations with Killing vector $\partial/\partial y^3$ (it is possible to construct solutions with non–Killing symmetries, see details in [7, 9]).

Let us define the values

$$\phi = \ln|\frac{\partial_4 h_3}{\sqrt{|h_3 h_4|}}|,\ \gamma := \partial_4 \ln\frac{|h_3|^{3/2}}{|h_4|},\ \alpha_i = \partial_4\sqrt{|h_3|}\partial_i\phi,\ \beta = \partial_4\sqrt{|h_3|}\partial_4\phi, \quad (23)$$

$$ {}^1\phi = \ln|\frac{\partial_6 h_5}{\sqrt{|h_5 h_6|}}|,\ {}^1\gamma := \partial_6 \ln\frac{|h_5|^{3/2}}{|h_6|},\ {}^1\alpha_\tau = \partial_6\sqrt{|h_5|}\partial_\tau\ {}^1\phi,\ {}^1\beta = \partial_6\sqrt{|h_5|}\partial_\tau\ {}^1\phi,$$

$$ {}^2\phi = \ln|\frac{\partial_8 h_7}{\sqrt{|h_7 h_8|}}|,\ {}^2\gamma := \partial_8 \ln\frac{|h_7|^{3/2}}{|h_8|},\ {}^2\alpha_{\tau_1} = \partial_8\sqrt{|h_7|}\partial_{\tau_1}\ {}^2\phi,\ {}^2\beta = \partial_8\sqrt{|h_7|}\partial_{\tau_1}\ {}^2\phi,$$

where $\phi(x^k,y^4)$, ${}^1\phi(u^\alpha,y^6,\theta)$ and ${}^2\phi(u^{\alpha_1},y^8,\theta)$ will be used as generating functions which can be re–defined by corresponding transforms of h_a, h_{a_1} and h_{a_2} (or inversely). A tedious computation of the N–adapted coefficients of the Ricci tensor of ${}^s\widehat{\mathbf{D}}$ for the ansatz (21) allows us to express (19) as

$$\begin{array}{rcl} \widehat{R}^1_1 &=& \widehat{R}^2_2 = -\Lambda(x^k),\ \widehat{R}^3_3 = \widehat{R}^4_4 = -\ {}^v\Lambda(x^k,y^4), \\ \widehat{R}^5_5 &=& \widehat{R}^6_6 = -\ {}^v_1\Lambda(u^\beta,y^6),\ \widehat{R}^7_7 = \widehat{R}^8_8 = -\ {}^v_2\Lambda(u^{\beta_1},y^8), \end{array} \quad (24)$$

with nontrivial effective (polarized gravitational constants) Λ–sources related to $\mathbf{\Upsilon}_{\beta_s\delta_s}$ (18) via formulas

$$\begin{array}{rcl} \mathbf{\Upsilon}^1_1 &=& \mathbf{\Upsilon}^2_2 = {}^v\Lambda + {}^v_1\Lambda + {}^v_2\Lambda, \mathbf{\Upsilon}^3_3 = \mathbf{\Upsilon}^4_4 = \Lambda + {}^v_1\Lambda + {}^v_2\Lambda, \\ \mathbf{\Upsilon}^5_5 &=& \mathbf{\Upsilon}^6_6 = \Lambda + {}^v_1\Lambda + {}^v_2\Lambda, \mathbf{\Upsilon}^7_7 = \mathbf{\Upsilon}^8_8 = \Lambda + {}^v\Lambda + {}^v_1\Lambda. \end{array}$$

For certain models of extra dimension gravity, and re–defining the integration functions, we can put ${}^v_1\Lambda = {}^v_2\Lambda = {}^\circ\Lambda = const.$

Using formulas (24), we can compute the Ricci scalar ${}^{sc}\widehat{R} := {}^s\widehat{\mathbf{R}}^{\beta_s}{}_{\beta_s}$, ${}^{sc}\widehat{R} = 2(\widehat{R}^1_1 + \widehat{R}^3_3 + \widehat{R}^5_5)$. As a result, we prove that there are certain N–adapted symmetries of the Einstein d–tensor $\widehat{E}\ {}_{\beta_s\delta_s} := {}^s\widehat{\mathbf{R}}\ {}_{\beta_s\delta_s} - \frac{1}{2}\mathbf{g}_{\beta_s\delta_s}\ {}^{sc}\widehat{R}$ for the ansatz (21) with Killing vector $\partial/\partial y^7$:

$$\begin{array}{rcl} s &=& 1 : \widehat{E}^1_1 = \widehat{E}^2_2 = -(\widehat{R}^3_3 + \widehat{R}^5_5), \widehat{E}^3_3 = \widehat{E}^4_4 = -(\widehat{R}^1_1 + \widehat{R}^5_5), \\ && \widehat{E}^5_5 = \widehat{E}^6_6 = -(\widehat{R}^1_1 + \widehat{R}^3_3), \\ s &=& 2 :\ \widehat{E}^1_1 = \widehat{E}^2_2 = -(\widehat{R}^3_3 + \widehat{R}^5_5 + \widehat{R}^7_7), \widehat{E}^3_3 = \widehat{E}^4_4 = -(\widehat{R}^1_1 + \widehat{R}^5_5 + \widehat{R}^7_7), \\ && \widehat{E}^5_5 = \widehat{E}^6_6 = -(\widehat{R}^1_1 + \widehat{R}^3_3 + \widehat{R}^7_7), \widehat{E}^7_7 = \widehat{E}^8_8 = -(\widehat{R}^1_1 + \widehat{R}^3_3 + \widehat{R}^5_5). \end{array}$$

Such symmetries of the linear connection ${}^s\widehat{\mathbf{D}}$ which can be nonholonomically constrained to ∇ are important for decoupling and constructing generic off–diagonal solutions of gravitational field equations.

3.3 Decoupling and integration of gravitational field eqs

The ansatz (21) for $g_i(x^k) = \epsilon_i e^{\psi(x^k)}, \epsilon_i = 1$, with nonzero $\partial_4\phi, \partial_4 h_a$, $\partial_6\ {}^1\phi, \partial_6 h_{a_1}, \partial_8\ {}^2\phi, \partial_8 h_{a_2}, ...$ transforms (19) into such a system of PDEs:

$$\epsilon_1\partial_{11}\psi + \epsilon_2\partial_{22}\psi = 2\Lambda(x^k), \quad (25)$$
$$\partial_4\phi\ \partial_4 h_3 = 2h_3h_4\ {}^v\Lambda(x^k, y^4), \quad (26)$$
$$\partial_{44}n_i + \gamma\partial_4 n_i = 0,\ \beta w_i - \alpha_i = 0,$$

$$\partial_6\ {}^1\phi\ \partial_6 h_5 = 2h_5h_6\ {}^v_1\Lambda(u^\beta, y^6, \theta), \quad (27)$$
$$\partial_{66}\ {}^1n_\tau +\ {}^1\gamma\partial_6\ {}^1n_\tau = 0,\ {}^1\beta\ {}^1w_\tau -\ {}^1\alpha_\tau = 0,$$

$$\partial_8\ {}^2\phi\ \partial_6 h_7 = 2h_7h_8\ {}^v_2\Lambda(u^{\beta_1}, y^8, \theta), \quad (28)$$
$$\partial_{88}\ {}^2n_{\tau_1} +\ {}^2\gamma\partial_8\ {}^2n_{\tau_1} = 0,\ {}^2\beta\ {}^2w_{\tau_1} -\ {}^2\alpha_{\tau_1} = 0.$$

Let us explain the decoupling property of (25)–(28): The equation in (25) is just the 2–de Laplace/Poisson equation which can be solved for any given $\Lambda(x^k)$. The first equation in the group (26) allows us to express h_3 through h_4 (or inversly) for any given generating function ϕ and nontrivial source ${}^v\Lambda$, see details in [7, 9]. Having h_3 and h_4, we can compute the coefficients γ, α_i, β, in (23) and find n_i and w_i, respectively, integating two times on y^4 and solving an algebraic equations. In a similar form, we can construct solutions for h_5 and h_6 and then for ${}^1n_\tau$ and ${}^1w_\tau$ for any generating function ${}^1\phi$ and source ${}^v_1\Lambda$ in the group (27). The same procedured can be used for constructing solutions for h_7 and h_7 and then for ${}^1n_{\tau_1}$ and ${}^1w_{\tau_1}$ because the system (28) is a similar extension of (27) on variable y^8.

We generate solutions of (19) and LC–conditions (20) by any

$$\begin{aligned} ds^2_K &= \epsilon_i e^{\psi(x^k,\mathring{\mu})}(dx^i)^2 + \frac{\tilde{\Phi}^2}{4\tilde{\Lambda}}\left[dy^3 + (\partial_i\ n)dx^i\right]^2 + \frac{(\partial_4\ \tilde{\Phi})^2}{\tilde{\Lambda}\ \Phi^2}\left[dy^4 + (\partial_i\ \check{A})dx^i\right]^2 \\ &+ \frac{{}^1\tilde{\Phi}^2}{4\ {}^1\tilde{\Lambda}}\left[dy^5 + (\partial_\tau\ {}^1n)du^\tau\right]^2 + \frac{(\partial_6\ {}^1\tilde{\Phi})^2}{{}^1\tilde{\Lambda}\ {}^1\tilde{\Phi}^2}\left[dy^6 + (\partial_\tau\ {}^1\check{A})du^\tau\right]^2 \quad (29) \\ &+ \frac{{}^2\tilde{\Phi}^2}{4\ {}^2\tilde{\Lambda}}\left[dy^7 + (\partial_{\tau_1}\ {}^2n)du^{\tau_1}\right]^2 + \frac{(\partial_8\ {}^2\tilde{\Phi})^2}{{}^2\tilde{\Lambda}\ {}^2\tilde{\Phi}^2}\left[dy^8 + (\partial_{\tau_1}\ {}^2\check{A})du^{\tau_1}\right]^2, \end{aligned}$$

where the generating functions $\Phi := e^\phi,\ {}^1\Phi := e^{\ {}^1\phi}$ and ${}^2\Phi := e^{\ {}^2\phi}$ are re–parameterized, respectively, $\Phi \to \tilde{\Phi},\ {}^1\Phi \to\ {}^1\tilde{\Phi},\ {}^2\Phi \to\ {}^2\tilde{\Phi}$, following

$$\begin{aligned} \frac{\partial_4[\Phi^2]}{{}^v\Lambda} &= \frac{\partial_4[\tilde{\Phi}^2]}{\tilde{\Lambda}}, \frac{\partial_6[\ {}^1\Phi^2]}{{}^v_1\Lambda} = \frac{\partial_6[\ {}^1\tilde{\Phi}^2]}{{}^1\tilde{\Lambda}}, \frac{\partial_4[\ {}^2\Phi^2]}{{}^v_2\Lambda} = \frac{\partial_4[\ {}^2\tilde{\Phi}^2]}{{}^2\tilde{\Lambda}}, \\ \tilde{\Lambda} &= const,\ {}^1\tilde{\Lambda} = const,\ {}^2\tilde{\Lambda} = const. \quad (30) \end{aligned}$$

The coefficients of metrics of type (29) satisfy the conditions

$$\begin{array}{ll} s=0: & \tilde{\Phi}=\check{\Phi}(x^{i},y^{4},\mathring{\mu}),\partial_{4}\partial_{k}\ \check{\Phi}=\partial_{k}\partial_{4}\ \check{\Phi}; \\ & \partial_{k}\ \check{\Phi}/\partial_{4}\ \check{\Phi}=\partial_{k}\ \check{A};\ {}_{1}n_{k}=\partial_{k}\ n(x^{i}); \\ s=1: & {}^{1}\tilde{\Phi}=\ {}^{1}\check{\Phi}(u^{\tau},y^{6},\mathring{\mu},\theta),\partial_{6}\partial_{\tau}\ {}^{1}\check{\Phi}=\partial_{\tau}\partial_{6}\ {}^{1}\check{\Phi}; \\ & \partial_{\alpha}\ {}^{1}\check{\Phi}/\partial_{6}\ {}^{1}\check{\Phi}=\partial_{\alpha}\ {}^{1}\check{A};\ {}_{1}^{1}n_{\tau}=\partial_{\tau}\ {}^{1}n(u^{\beta},\mathring{\mu},\theta); \\ s=2: & {}^{2}\tilde{\Phi}=\ {}^{2}\check{\Phi}(u^{\tau_{1}},y^{8},\mathring{\mu},\theta),\partial_{8}\partial_{\tau_{1}}\ {}^{2}\check{\Phi}=\partial_{\tau_{1}}\partial_{8}\ {}^{2}\check{\Phi}; \\ & \partial_{\alpha_{1}}\ {}^{2}\check{\Phi}/\partial_{8}\ {}^{2}\check{\Phi}=\partial_{\alpha_{1}}\ {}^{2}\check{A};\ {}_{1}^{2}n_{\tau_{1}}=\partial_{\tau_{1}}\ {}^{2}n(u^{\beta_{1}},\mathring{\mu},\theta). \end{array} \tag{31}$$

The values $n(x^{i},\mathring{\mu},)$, ${}^{1}n(u^{\beta},\mathring{\mu},\theta)$ and ${}^{2}n(u^{\beta_{1}},\mathring{\mu},\theta)$ are integrating functions which together with $\check{A}(x^{i},y^{4},\mathring{\mu},)$, ${}^{1}\check{A}(u^{\tau},y^{6},\mathring{\mu},\theta)$ and ${}^{2}\check{A}(u^{\tau_{1}},y^{8},\mathring{\mu},\theta)$ determine the N–connection coefficients (22).

If conditions of type (30) are not imposed, we can consider instead of $\partial_{i}\ \check{A},\partial_{\tau}\ {}^{1}\check{A},\partial_{\tau}\ {}^{1}\check{A}$, respectively, the values

$$\begin{array}{rcl} w_{i} &=& \partial_{i}\tilde{\phi}/\partial_{4}\widetilde{\phi}=\partial_{i}\tilde{\Phi}/\partial_{4}\tilde{\Phi},\ {}^{1}w_{\tau}=\partial_{\tau}\ {}^{1}\phi/\partial_{6}\ {}^{1}\phi=\partial_{\tau}\ {}^{1}\tilde{\Phi}/\partial_{4}\ {}^{1}\tilde{\Phi}, \\ {}^{2}w_{\tau_{1}} &=& \partial_{\tau_{1}}\ {}^{2}\phi/\partial_{8}\ {}^{2}\phi=\partial_{\tau_{1}}\ {}^{2}\tilde{\Phi}/\partial_{8}\ {}^{2}\tilde{\Phi}, \end{array}$$

and construct exact solutions of (19) which do not solve the LC–conditions (20). Such configurations are with nonholonomically induced noncommutative torsion on $s=1,2$ shells.

The values (31) contain dependence on integration parameters, for instance, on a "commutative" one which can be related to the mass of graviton $\mathring{\mu}$ and on "noncommutative" ones θ induced by possible Schrödinger type relations of type (3). If the LC–conditions (20) are satisfied, the solutions of type (29) are for effective Einstein spaces with generic off–diagonal metrics and polarizations of constants determined by massive gravity, modifications of Lagrangians and certain noncommutative θ–deformations. Such solutions may mimic violation of Lorentz symmetries and HL–type anisotropy.

4 On Renormalizable Modified Massive Gravity

4.1 Mimicking modified gravity and non–standard perfect fluid coupling

The solutions for theories for ${}^{[1]}\mathcal{L}$ and ${}^{[2]}\mathcal{L}$ are equivalent for classes of generating functions and sources satisfying the conditions (30). Various models of modified gravity with possible massive terms and bimetric/ biconnection structures can be transformed into off–diagonal configurations of certain effective Einstein spaces with shell cosmological constants $\widetilde{\Lambda}$, ${}^{1}\tilde{\Lambda}$, ${}^{2}\tilde{\Lambda}$.

The next step is to state the conditions when the solutions for a theory ${}^{[2]}\mathcal{L}$ are encoded into a model ${}^{[3]}\mathcal{L}$. This allows us to elaborate certain analogies of modified gravity to theories with non–standard perfect fluid coupling [3]. We work with generic off–diagonal metrics (11) in 4–d (the

constructions on (non) commutative fibers can be performed via canonical lifts from the commutative spacetime). With respect to coordinate frames and for a flat background metric $\eta_{\alpha\beta}$, we write any $\mathbf{g}_{\alpha\beta}$ (11) as a generic off–diagonal metric $^{\diamond}\mathbf{g}_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}(x^i, t)$, where $y^4 = t$ is the timelike coordinate. Choosing the "gauge" conditions $h_{tt} = h_{t\widehat{i}} = h_{\widehat{i}t} = 0$, for $\widehat{i}, \widehat{j}, = 1, 2, 3$ for a "double" splitting $(3+1)$ and $(2+2)$ on a manifold $\mathbf{V}$, we express the corresponding Ricci tensor and scalar curvature in the form

$$
\begin{aligned}
{}^{\diamond}R_{\widehat{i}\widehat{j}} &= \frac{1}{2}(h^{**}_{\widehat{i}\widehat{j}} + \partial_{\widehat{i}}\partial^{\widehat{k}} h_{\widehat{j}\widehat{k}} + \partial_{\widehat{j}}\partial^{\widehat{k}} h_{\widehat{i}\widehat{k}} - \partial_{\widehat{k}}\partial^{\widehat{k}} h_{\widehat{i}\widehat{j}}),\ {}^{\diamond}R_{44} = -\frac{1}{2}\delta^{\widehat{i}\widehat{j}} h^{**}_{\widehat{i}\widehat{j}};\\
{}^{\diamond}R &= \delta^{\widehat{i}\widehat{j}}(h^{**}_{\widehat{i}\widehat{j}} - \partial_{\widehat{k}}\partial^{\widehat{k}} h_{\widehat{i}\widehat{j}}) + \partial^{\widehat{i}}\partial^{\widehat{j}} h_{\widehat{i}\widehat{j}}.
\end{aligned}
$$

For a prescribed generating function $\widetilde{\phi}(x^i, t), \partial_t \widetilde{\phi} \neq 0$, there are elaborated models with $^{[2]}\mathcal{L} = {}^{s}\widetilde{R} + \widetilde{L} = {}^{[3]}\mathcal{L} = {}^{\diamond}R + {}^{\diamond}L$, with $^{\diamond}L$ taken for $^{\diamond}\mathbf{g}_{\alpha\beta}$ and an effective non–standard coupling with a fluid configuration. Einstein manifolds encoding fluid like configurations are generated if $\widetilde{\phi}$ contains parameters $\breve{\alpha}, \breve{\beta}, \rho, \varpi$ introduced into equations

$$
\begin{aligned}
\frac{\partial_t \widetilde{\phi}\ \partial_t h_4}{2 h_3 h_4} &= -\,{}^{\diamond}L = \breve{\alpha}\rho^2\{[\breve{\beta}(3\varpi - 1) + \frac{\varpi - 1}{2}]\delta^{\widetilde{\widehat{i}\widehat{j}}} h^{**}_{\widetilde{\widehat{i}\widehat{j}}} + \\
& (\varpi + 3\varpi\breve{\beta} - \breve{\beta})(\partial^{\widehat{i}}\partial^{\widehat{j}} h_{\widetilde{\widehat{i}\widehat{j}}} - \partial_{\widehat{k}}\partial^{\widehat{k}} h_{\widetilde{\widehat{i}\widehat{j}}})\}^2. \qquad (32)
\end{aligned}
$$

We write constants $\breve{\alpha}$ and $\breve{\beta}$ instead of α and β in [3] to avoid possible ambiguities related to coefficients α and β from (23). In above formulas, we wrote $^{\diamond}\Upsilon = -\,{}^{\diamond}L$ in order to emphasize that such a source is determined for a metric $^{\diamond}\mathbf{g}_{\alpha\beta}$ and Lagrangian $^{\diamond}L$. Using this expression and formulas (29) redefined following (30), we find $h_3[\tilde{\Phi}] = e^{2\widetilde{\phi}}/4\widetilde{\Lambda}$ and $h_4[\tilde{\Phi}] = (\partial_4\widetilde{\phi})^2/\widetilde{\Lambda}$, which for $\tilde{\Phi} = e^{\widetilde{\phi}},\ \widetilde{\Lambda}e^{2\widetilde{\phi}} = e^{2\ {}^{0}\widetilde{\phi}} \int dt|\ {}^{\diamond}\Upsilon|\partial_t\left(e^{2\phi}\right)$ and $w_i = \partial_i\phi/\partial_t\phi$ can be used for a $^{[1]}\mathcal{L}$–theory.

In the limit $\breve{\beta} \to (1-\varpi)/2(3\varpi - 1)$, we can express $^{\diamond}\Upsilon = \breve{\alpha}\frac{\rho^2}{4}(\varpi + 1)^2[(\partial^{\widehat{i}}\partial^{\widehat{j}} h_{\widetilde{\widehat{i}\widehat{j}}} - \partial_{\widehat{k}}\partial^{\widehat{k}} h_{\widetilde{\widehat{i}\widehat{j}}})]^2$ and generate a class of Einstein manifolds

$$
\begin{aligned}
\mathbf{g} &= \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + \epsilon_3 \int dt |4\ {}^{\diamond}\Upsilon|^{-1}\partial_t(e^{2\widetilde{\phi}})\mathbf{e}^3 \otimes \mathbf{e}^3 \\
& + \epsilon_4 \left[\partial_t\left(\sqrt{\left|\int dt|\ {}^{\diamond}\Upsilon|^{-1}\partial_t(e^{2\widetilde{\phi}})\right|}\right)\right]^2 e^{-2\widetilde{\phi}}\mathbf{e}^4 \otimes \mathbf{e}^4, \qquad (33)\\
\mathbf{e}^3 &= dy^3 + (\partial_i\phi/\partial_4\phi)dx^i,\ \mathbf{e}^4 = dy^4 + n_i dx^i,
\end{aligned}
$$

where $\epsilon_\alpha = \pm 1$ depending on signature. Such manifolds are with Killing symmetry on $\partial/\partial y^3$ and broken Lorentz invariance because the source $^{\diamond}\Upsilon$ does not contain the derivative with respect to $\partial/\partial y^3$.

It is not surprising that off–diagonal solutons of type (33) are not Lorentz invariant. For small nonholonomic deformations we can model, for instance,

rotoid Schwarzschild - de Sitter configurations which are diffeomorphysm invariant and with broken Lorentz symmetry [7]. In the ultraviolet region with large momentum $\mathbf{k}$, the second term for the equivalent theory ${}^{[3]}\mathcal{L}$ gives the propagator $|\mathbf{k}|^{-4}$. For such configurations, the longitudinal modes do not propagate being allowed propagation of the transverse one with possible polarization of constant and additional off–diagonal terms. Alternatively, we can say that we elaborated a theory with non–standard coupling of modified/ massive gravity with perfect fluid when the energy–momentum tensor $T_{\widehat{i}\widehat{j}} = p\delta_{\widehat{i}\widehat{j}} = \varpi\rho\delta_{\widehat{i}\widehat{j}}$ and $T_{44} = \rho$. Treating p, ρ and ϖ as standard fluid parameters and the equation of state in the flat background we compute ${}^{\diamond}L = -\alpha(T^{\alpha\beta}\ {}^{\diamond}R_{\alpha\beta} + \beta T^{\alpha}_{\alpha}\ {}^{\diamond}R^{\beta}_{\beta})^2$. The effective Ricci tensor can be considered for ∇, $\widehat{\mathbf{D}}$ or any other metric compatible connection completely defined by the matric structure following certain well defined geometric principles.

The off–diagonal solutions for theories with ${}^{[1]}\mathcal{L}$ and/or ${}^{[2]}\mathcal{L}$ derived for generating functions and effective sources of type (24) encode a kind of spontaneous violation of symmetry which is typical in quantum field theories and condensed matter models. For corresponding nonholonomic configurations, we elaborate configurations vacuum gravitational aether via analogous coupling with non–standard fluid which breaks the Lorentz symmetry for an effective equivalent theory ${}^{[3]}\mathcal{L}$ and mimic massive gravity contributions. This allows us to elaborate a (power–counting) renormalizable model of QG, or to consider other quantization procedures like deformation quantization, A–brane quantization, gauge like gravity quantum models etc [8].

4.2 Effective renormalizable (non)commutative modified and massive gravity

In this section, we consider a flat background approximation in order to study equivalence of modified gravity theories (for certain classes of nonholonomic constraints) to certain models with non–standard fluid coupling. Such constructions can be extended to curved (in general, noncommutative tangent bundles) backgrounds and generic off–diagonal gravitational–field interactions using solutions of type (29). Using generalizations of the principle of relativity [9] (for the so–called Einstein–Finsler models, noncommutative gravity etc), we can work in a local N–adapted Lorentz frame when, for instance, the effective fluid does not flow. This allows us to preserve unitarity and formulate certain generalize axioms as in GR working with general class of solutions for theories ${}^{[1]}\mathcal{L}$ and/or ${}^{[2]}\mathcal{L}$. Choosing corresponding classes of generating functions with necessary type parametric dependence via corresponding nonholonomic transforms we mimic some models ${}^{[3]}\mathcal{L}$ with anisotropic coupling. The solutions of type (29) and (33) in the diagonal spherical symmetry limit (here we include the condition $T^{\alpha\beta} = 0$) contain the Schwarzschild and Kerr black hole/ellipsoid metrics with various tangent bundle extensions etc, see details in [7]. Such configurations with

$\Upsilon = {}^{\diamond}\Upsilon = -{}^{\diamond}L$ (32) result in $z=2$ Hořava–Lifshitz theories with canonical (non) commutative extensions on tangent Lorentz bundles.

The Lagrangian densities (1) (for simplicity, we consider here 4–d theories) include $z=3$ theories which allows us to generate ultra–violet power counting renormalizable $3+1$ and/or $2+2$ quantum models. We can take more general sources and generating functions instead of ${}^{\diamond}L$ and write

$$-\ {}^{\diamond}_{\hat{n}}L = \hat{\alpha}\{(T^{\mu\nu}\ {}^{\diamond}\nabla_{\mu}\ {}^{\diamond}\nabla_{\nu} + \hat{\gamma}T^{\alpha}_{\alpha}\ {}^{\diamond}\nabla^{\beta}\ {}^{\diamond}\nabla_{\beta})^{\hat{n}}(T^{\alpha\beta}\ {}^{\diamond}R_{\alpha\beta} + \hat{\beta}T^{\alpha}_{\alpha}\ {}^{\diamond}R^{\beta}_{\beta})\}^{2}, \quad (34)$$

where $\hat{n}$ and $\hat{\gamma}$ are constants and ${}^{\diamond}\nabla_{\mu}$ is obtained via nonholonomic constraints of any ${}^{\diamond}\mathbf{D}_{\mu}$ which is metric compatible and completely determined by a metric structure. Such configurations are determined by modified generating functions as following. We use $\Upsilon = -\ {}^{\diamond}_{\hat{n}}L$ in formulas (32) which results in a different class of generating functions ${}^{\hat{n}}\widetilde{\phi}$ from $\frac{\partial_4({}^{\hat{n}}\widetilde{\phi})\partial_4 h_3}{2h_3h_4} = -\ {}^{\diamond}_{\hat{n}}L$. A new class of off–diagonal solutions (33) can be generated for data $\widetilde{\phi} \to {}^{\hat{n}}\widetilde{\phi}, \phi \to {}^{\hat{n}}\phi$ and ${}^{\diamond}\Upsilon \to -\ {}^{\diamond}_{\hat{n}}L$. Twisted (non) commutative extensions to higher "velocity/ momentum" type fibers can be performed in a standard manner as we considered in the previous section. The formalism allows us to work with models (34) when $\hat{n}$ are non–integer, for instance, $\hat{n} = 1/2, 3/2$ etc (we do not study such theories in our works). We can apply the analysis provided in [3, 8] which states that the analogous ${}^{[3]}\mathcal{L}$ and ${}^{[4]}\mathcal{L}$ models derived for (34) are renormalizable if $\hat{n} = 1$ and super–renormalizable for $\hat{n} = 2$. Such properties can be preserved for theories generalized on tangent Lorentz bundles.

Both on base spacetime manifold and on total space of corresponding Lorentz bundles, the values induced by nontrivial (effective) sources and a Lagrangian density ${}^{\diamond}_{\hat{n}}L$ contain higher derivative terms. Such terms break the Lorentz symmetry for high energies, i.e. in UV region (which follows explicitly from noncommutative configurations). In the IR limits, we can consider nonholonomic constraints when effective Einstein configurations can be generated. Working with off–diagonal nonlinear systems of PDE and their solutions, we can obtain the (effective) GR not only as limits from certain modifications with non-standard coupling of type (32) and/or (34). We can model possible "branches" of complexity, anisotropies, effective massive graviton contributions, inhomogeneities and Lorentz violations depending on parameters and generating functions. For well defined conditions, we can select families of solutions which are (super) renormalizable because of off–diagonal nonlinear interactions of gravitational and effective matter fields. The models are, in general, with noncommutative quantum corrections.

5 Concluding Remarks

Our results mean that using generic off–diagonal solutions for analogous (effective) gravity theories we can model various effects from modified grav-

ity, massive gravity and various commutative and extra dimension generalizations on tangent bundles. The main conclusion is that we can keep an "orthodox" physical paradigm when the classical and quantum gravity theories are maximally closed to Einstein gravity. Our opinion is that the bulk of experimental data for modern gravity and cosmology can be explained/ predicted using generic off–diagonal solutions in general relativity (GR) and certain quantized locally anisotropic versions on tangent Lorentz bundles. Such models can be encoded into nonholonomic Einstein manifolds and effective models of interactions with broken Lorentz symmetry and covariant renormalization.

In this paper, we do not prove explicitly the renormalizability of off–diagonal gravitational configurations but show that they can be associated/ related to certain "renormalizable" and ghost free quantum gravity models studied by other authors. The second main conclusion is that generic self–accelerating solutions in massive/ bigravity are anisotropic and can be alternatively modelled by modified gravity theories or, via nonholonomic constraints, as certain nonlinear off–diagonal interactions. Noncommutative models arise naturally on tangent Lorentz bundles if uncertainty quantum relations are included into consideration.

It would also be interesting to study off–diagonal deformations of Kerr black holes in modified massive gravity and higher dimensions theories with (non) commutative velocity/ momentum type variables. Yet another interesting direction of research would be to study off–diagonal ekpyrotic scenarios and possible equivalence of dark energy and dark matter models in modified, massive and/or (effective) Einstein gravity.

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